Suggested Solution to Assignment 1

1. (a) Since $27 = 11011_2$ and $17 = 10001_2$

Now
$$\begin{array}{r} 11011_2 \\ \oplus\, 10001_2 \\ \hline 1010_2 \end{array}$$

Thus $27 \oplus 17 = 1010_2 = 10$

(b) It follows from $x \oplus 38 = 25$ that
$$\begin{aligned} x &= x \oplus 38 \oplus 38 \\ &= 25 \oplus 38 \\ &= 11001_2 + 100110_2 \\ &= 111111_2 \\ &= 63 \end{aligned}$$

(c) If $x \oplus y \oplus z = 0$, then
$$x \oplus y = x \oplus y \oplus z \oplus z = 0 \oplus z = z$$

Thus $x = x \oplus y \oplus y = z \oplus y = y \oplus z$

2. (a) Since $29 = 11101_2$, $20 = 10100$ and $15 = 1111_2$

Now
$$\begin{array}{r} 11101_2 \\ 10100_2 \\ \oplus\quad 1111_2 \\ \hline 110_2 \end{array}$$

Thus $29 \oplus 20 \oplus 15 = 110_2 = 6$

(b) We have $29 \oplus 20 \oplus 15 = 11101_2 \oplus 10100_2 \oplus 1111_2 = 110_2 = 6$

By looking at the leftmost column, the winning move is to reach 11011_2 from 11101_2 or 10010_2 from 10100_2 or 1001_2 from 1111_2.

Thus, all winning moves are $(27, 20, 15)$, $(29, 18, 15)$ and $(29, 20, 9)$.

3. (a) Since $12 = 1100_2$, $19 = 10011_2$ and $27 = 11011_2$

We have $12 \oplus 19 \oplus 27 = 1100_2 \oplus 10011_2 \oplus 11011_2 = 100_2 = 4$

The only winning move is to take 4 from 12 since it is the only number 1 at the leftmost column.

Thus, the winning move is $(8, 19, 27)$

(b) Since $13 = 1101_2$, $17 = 10001_2$, $19 = 10011_2$ and $23 = 10111_2$

We have $13 \oplus 17 \oplus 19 \oplus 23 = 1101_2 \oplus 10001_2 \oplus 10011_2 \oplus 10111_2 = 11000_2 = 24$

By looking at the leftmost column, the winning moves are

$(13, 9, 19, 23)$, $(13, 17, 11, 23)$ and $(13, 17, 19, 15)$.

4. (a) $P = \{x \in \mathbb{N} : x \equiv 0, 2 \pmod 8\}$ is the set of P-positions of the game.

(b) (i) The only terminal position is 0 and $0 \in P$.

(ii) Take any position $p \in P$. There are two cases: (1) when $p \equiv 0 \pmod 8$, it can be moved to a position q where $q \equiv 3, 4, 5, 7 \pmod 8$, which shows that $q \notin P$; (2) when $p \equiv 2 \pmod 8$, it can be moved to a position q where $q \equiv 1, 5, 6, 7 \pmod 8$, which implies also that $q \notin P$. In these two cases, any move from any position $p \in P$ reaches a position $q \notin P$.

(iii) Suppose there are q chips in the pile where $q \notin P$. For $q \equiv 1, 3, 4, 5, 6, 7 \pmod 8$, the nest player may remove $1, 3, 4, 5, 4, 5$ chips such that the remaining chips are congruent to $0, 0, 0, 0, 2, 2$ modulo 8 respectively and are positions in P. Therefore, for any position $q \notin P$, there exists a move from q reaching a position $p \in P$.

Thus, $P = \{x \in \mathbb{N} : x \equiv 0, 2 \pmod 8\}$ is the set of P-positions.

So g(4)=2, g(18)=0, g(29)=3

5. (a) $S = \{1, 2, 6\}$, we list the values of $g(x)$ below

x	0	1	2	3	4	5	6	7	8	9	10	11	12	$\cdots$
$g(x)$	0	1	2	0	1	2	3	0	1	2	0	1	2	$\cdots$

In fact, it can be shown that

$$g(x) = \begin{cases} 0, & \text{if } x \equiv 0, 3 \pmod 7 \\ 1, & \text{if } x \equiv 1, 4 \pmod 7 \\ 2, & \text{if } x \equiv 2, 5 \pmod 7 \\ 3, & \text{if } x \equiv 6 \pmod 7 \end{cases}$$

Since $4 \equiv 4 \pmod 7$, $6 \equiv 6 \pmod 7$, $100 \equiv 2 \pmod 7$

We have $g(4) = 1$, $g(6) = 3$ and $g(100) = 2$.

(b) By the hypothesis, the possible moves from 100 for the first player are 99, 98 and 94.

Since $99 \equiv 1 \pmod 7$, $98 \equiv 0 \pmod 7$, $94 \equiv 3 \pmod 7$.

From (a), we have $g(99)=1$, $g(98)=0$ and $g(94)=0$

Thus, 98 and 94 are P-positions, i.e. all winning moves are 98 and 94.

(c) $P = \{x \in \mathbb{N} : x \equiv 0, 3 \pmod 7\}$ is the set of P-positions of the game.

Pf. (i) The only terminal position is 0 and $0 \in P$.

(ii) Take any position $p \in P$. There are two case: (1) when $p \equiv 0 \pmod 7$, it can be moved to a position q where $q \equiv 1, 2, 6 \pmod 7$, which shows that $q \notin P$; (2) when $p \equiv 3 \pmod 7$, it can be moved to a position q where $q \equiv 4, 5, 2 \pmod 7$, which also implies that $q \notin P$.
In these two cases, any move from any position $p \in P$ reaches a position $q \notin P$.

(iii) Suppose there are q chips in the pile where $q \notin P$. For $q \equiv 1, 4, 2, 5, 6 \pmod 7$, the nest player may remove $1, 1, 2, 2, 6$ chips respectively such that the remaining chips are congruent to $0, 3, 0, 3, 0$ modulo 8 respectively and are positions in P. Therefore, for any position $q \notin P$, there exists a move from q reaching a position $p \in P$.

Thus, $P = \{x \in \mathbb{N} : x \equiv 0, 3 \pmod 7\}$ is the set of P-positions.

6. (a) List the first few P-positions of the game:

$(0,0), (1,2), (3,5), (4,7), (6,10), (8,13), (9,15), (11,18), \cdots$

Thus, for $(6,9)$ the winning move is $(4,7)$; for $(11,15)$, the winning moves are $(9,15)$ and $(6,10)$; for $(13,20)$, the winning moves are $(13,8)$, $(12,20)$ and $(11,18)$.

(b) Let $\varphi = \frac{1+\sqrt{5}}{2}$.

(i) Suppose $[n\varphi] = 100$, then we can take $n = 62$. Thus $[n\varphi] + n = 100 + 62 = 162$.

Hence $(x, y) = (100, 162)$.

(ii) If we take $[n\varphi] = 500$, then n has no solutions. Thus we must choose n such that $[n\varphi] + n = 500$. We can take $n = 191$, then $[n\varphi] = 309$ and $[n\varphi] + n = 500$.

Hence $(x, y) = (500, 309)$.

(iii) $x - y = 999 \Rightarrow ([n\varphi] + n) - [n\varphi] = 999$, so $n = 999$. Thus $[n\varphi] = 1616$ and $[n\varphi] + n = 2615$.

Hence $(x, y) = (2615, 1616)$.

8. (a) According to the lectures, the Sprague-Grundy functions for 1-pile nim and subtraction game with $S = \{1, 2, 3, 4, 5, 6\}$ are $g_1(x) = x$ and $g_2(x) \equiv x \pmod 7$ respectively. It remains to find the Sprague-Grundy function $g_3(x)$ for Game 3. We first list the values of $g_3(x)$ as follows:

x	0	1	2	3	4	5	6	7	8	9	10 ⋯
$g_3(x)$	0	1	1	0	2	0	3	0	4	0	5 ⋯

In fact, it is not difficult to see that

$$g_3(x) = \begin{cases} 0, & \text{if } x \text{ is odd and } x \neq 1 \\ \frac{x}{2}, & \text{if } x \text{ is even} \\ 1, & \text{if } x = 1 \end{cases}.$$

Hence, $g_1(14) = 14$, $g_2(17) = 3$ and $g_3(24) = \frac{24}{2} = 12$.

(b) Since $14 = 1110_2$, $3 = 11_2$ and $12 = 1100_2$

$$\begin{aligned} g(14, 17, 24) &= g_1(14) \oplus g_2(17) \oplus g_3(24) \\ &= 14 \oplus 3 \oplus 12 \\ &= 1110_2 \oplus 11_2 \oplus 1100_2 \\ &= 1_2 \\ &= 1 \end{aligned}$$

(c) Let the winning move of G from the position $(14, 17, 24)$ be (x, y, z).

We can make a move in exactly one of the Game 1, 2, 3 (while the other two remain unchanged) such that $g_1(x) = 1111_2 = 15$, $g_2(y) = 10_2 = 2$ or $g_3(z) = 1101_2 = 13$. Note that since the number of chips is decreasing in succeeding moves, the case $g_1(x) = 15$ is impossible.

For $g_2(y) = 2$, we may take $y = 16$

For $g_3(z) = 13$, it is impossible too because the number of chips is decreasing in succeeding moves.

Hence, there is only one winning move from the position $(14, 17, 24)$ to $(14, 16, 24)$. (i.e. remove 1 chip in Game 2).

9. (a) According to the lectures, the Sprague-Grundy functions for 1-pile nim and subtraction game with $S = \{1, 2, 3, 4, 5, 6, 7\}$ are $g_1(x) = x$ and $g_2(x) \equiv x \pmod 8$ respectively. It remains to find the Sprague-Grundy function $g_3(x)$ for Game 3.

We first list the values of $g_3(x)$ as follows:

x	0	1	2	3	4	5	6	7	8	9	10 ...
$g_3(x)$	0	1	2	1	0	1	2	1	0	1	2

In fact, it is not difficult to see that

$$g_3(x) = \begin{cases} 0, & \text{if } x \equiv 0 \pmod 4 \\ 1, & \text{if } x \equiv 1, 3 \pmod 4 \\ 2, & \text{if } x \equiv 2 \pmod 4 \end{cases}.$$

Hence, $g_1(7) = 7$, $g_2(14) = 6$, $g_3(18) = 2$.

(b) Since $7 = 111_2$, $6 = 110_2$, and ~~8~~ $2 = 10_2$.

$$\begin{aligned} g(7, 14, 18) &= g_1(7) \oplus g_2(14) \oplus g_3(18) \\ &= 7 \oplus 6 \oplus 2 \\ &= 111_2 \oplus 110_2 \oplus 10_2 \\ &= 11_2 \\ &= 3 \end{aligned}$$

(c) Let the winning move of G from the position $(7, 14, 18)$ be (x, y, z).

We can make a move in exactly one of the Game 1, 2, 3 such that

$g_1(x) = 100_2 = 4$, $g_2(y) = 101_2 = 5$ or $g_3(z) = 1_2 = 1$.

For $g_1(x) = 4$, we may take $x = 4$.

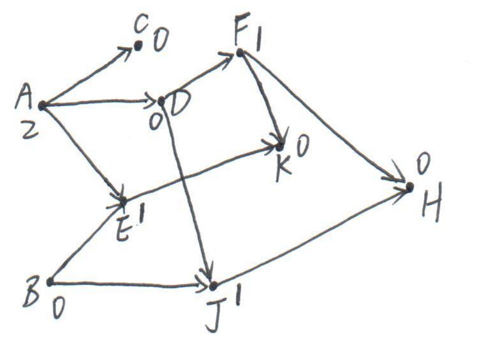
For $g_2(y) = 5$, we may take $y = 13$.

For $g_3(z) = 1$, we may take $z = 17$.

Hence, all winning moves of G from the position $(7, 14, 18)$ are

$(4, 14, 18)$, $(7, 13, 18)$ and $(7, 14, 17)$.

10. (a)



(b) By (a), all vertices which are at P-position but not at terminal position are:

B and D.

(c) (i) Since $g_1(A) = 2$, $g_2(B) = 0$, $g_3(E) = 1$

We have
$$\begin{aligned} g(A, B, E) &= g_1(A) \oplus g_2(B) \oplus g_3(E) \\ &= 2 \oplus 0 \oplus 1 \\ &= 10_2 \oplus 1_2 \\ &= 11_2 \\ &= 3 \end{aligned}$$

(ii) Let the winning move from (A, B, E) be (x, y, z). We can make a move in exactly one of the three games such that $g_1(x) = 01_2 = 1$, $g_2(y) = 11_2 = 3$, or $g_3(z) = 10_2 = 2$.

Note that since the moves from A contain C, D, E; from B contain E, J; from E contains K.

For $g_1(x) = 1$, we may take $x = E$. For $g_2(y) = 3$ or $g_3(z) = 2$, they are impossible.

Hence, the winning move from (A, B, E) is (E, B, E).

11. (a) According to the lectures, the Sprague-Grundy function of the take-and-break game is given by

$$g(x) = \begin{cases} 4k+1, & \text{if } x = 4k+1 \\ 4k+2, & \text{if } x = 4k+2 \\ 4k+4, & \text{if } x = 4k+3 \\ 4k+3, & \text{if } x = 4k+4 \end{cases}$$

Since $10 = 8+2$, $11 = 8+3$ and $12 = 8+4$

we have $g(10) = 10$, $g(11) = 12$, $g(12) = 11$.

(b)
$$\begin{aligned} g(10,11,12) &= g(10) \oplus g(11) \oplus g(12) \\ &= 10 \oplus 12 \oplus 11 \\ &= 1010_2 \oplus 1100_2 \oplus 1011_2 \\ &= 1101_2 \\ &= 13 \end{aligned}$$

Thus the next player has winning strategies moving to positions with Sprague-Grundy values either $(111_2, 12, 11)$, $(10, 1_2, 11)$ or $(10, 12, 110_2)$, i.e. $(7, 12, 11)$, $(10, 1, 11)$ or $(10, 12, 6)$.

For $(7, 12, 11)$, the player can remove 2 chips from the first pile to reach $(8, 11, 12)$.

For $(10, 1, 11)$, the player can remove 10 chips from the second pile to reach $(10, 1, 12)$.

For $(10, 12, 6)$, the player can remove 6 chips from the third pile to reach $(10, 11, 6)$.

Therefore, all winning moves from $(10, 11, 12)$ are:

$(8, 11, 12)$, $(10, 1, 12)$ and $(10, 11, 6)$.

Supplementary:

4. (c) $S = \{1, 3, 4, 5\}$, we list the values of $g(x)$ below

x	0	1	2	3	4	5	6	7	8	9	10	11	12	⋯
$g(x)$	0	1	0	1	2	3	2	3	0	1	0	1	2	⋯

It can be shown that

$$g(x) = \begin{cases} 0, & \text{if } x \equiv 0, 2 \pmod 8 \\ 1, & \text{if } x \equiv 1, 3 \pmod 8 \\ 2, & \text{if } x \equiv 4, 6 \pmod 8 \\ 3, & \text{if } x \equiv 5, 7 \pmod 8 \end{cases}$$

Since $4 \equiv 4 \pmod 8$, $18 \equiv 2 \pmod 8$, $29 \equiv 5 \pmod 8$,

we have $g(4) = 2$, $g(18) = 0$ and $g(29) = 3$.

7: (a) According to the lectures, the set of P-positions of the ordinary nim game is

$P_0 = \{(x_1, x_3, x_5) : x_1 \oplus x_3 \oplus x_5 = 0\}$

Now we need to show $P = \{(x_1, x_2, x_3, x_4, x_5) : x_1 \oplus x_3 \oplus x_5 = 0\}$ is the set of P-positions of the staircase nim game.

Check the 3 conditions for P-positions as follows.

(i) The only terminal position is $(0,0,0,0,0)$ and $(0,0,0,0,0) \in P$.

(ii) Take any position $(x_1, x_2, x_3, x_4, x_5) \in P$. Then $x_1 \oplus x_3 \oplus x_5 = 0$. Suppose the position $(x_1, x_2, x_3, x_4, x_5)$ is moved to $(x_1', x_2', x_3', x_4', x_5')$. We know only one of x_1, x_3, x_5 is changed. Without loss of generality, we may assume $x_1' < x_1$ or $x_1' > x_1$, $x_3' = x_3$, $x_5' = x_5$.

Now if $x_1' \oplus x_3' \oplus x_5' = 0$, then by cancelation law,

$x_1' \oplus x_3 \oplus x_5 = x_1 \oplus x_3 \oplus x_5 \Rightarrow x_1' = x_1$

which is a contradition. Thus a position in P cannot move to a position in P.

(iii) Suppose $(x_1, x_2, x_3, x_4, x_5) \notin P$. Then $x_1 \oplus x_3 \oplus x_5 \neq 0$. When calculating the nim sum using the binary expressions of x_1, x_3, x_5, look at the leftmost column with odd number of 1. Choose a pile that the corresponding binary digit of the number of chips in the pile is 1. Remove the chips from the pile for the ~~fifth~~ pile or remove the chips from the second or forth pile to the first or third pile respectively so that all columns contain even number of 1 and the nim-sum of 3 numbers would become 0. Therefore any position not in P has a move to a position in P.

Thus, $(x_1, x_2, x_3, x_4, x_5)$ is a P position if and only if (x_1, x_3, x_5) is a P-position in ordinary nim.

(b) From (a), since $4 = 100_2$, $9 = 1001_2$, $14 = 1110_2$, we have

$4 \oplus 9 \oplus 14 = 100_2 \oplus 1001_2 \oplus 1110_2 = 11_2 = 3$

By looking at the leftmost column, the winning move is to reach 111_2 from 100_2, 1010_2 from 1001_2 or 1101_2 from 1110_2.

Thus, all winning moves from the position $(4, 6, 9, 11, 14)$ are:

$(7, 3, 9, 11, 14)$, ~~(4, 7, 8, 11, 14)~~ $(4, 6, 10, 10, 14)$ and $(4, 6, 9, 12, 13)$.

12.

a.

$\Rightarrow$ Denote $\lfloor n\alpha \rfloor = k$. then we have

$$0 < n\alpha - k < 1$$

$$k < n\alpha < k+1$$

$$\frac{k}{\alpha} < n < \frac{k+1}{\alpha}$$

Since $\frac{k+1}{\alpha} > n$, then $\lfloor \frac{k}{\alpha} \rfloor \geq n-1$

Since $\frac{k}{\alpha} < n$, then $\lfloor \frac{k}{\alpha} \rfloor \leq n-1$

Then $\lfloor \frac{k}{\alpha} \rfloor = n-1$

Then $\frac{k+1}{\alpha} - \lfloor \frac{k}{\alpha} \rfloor > 1$

$$\frac{k}{\alpha} - \lfloor \frac{k}{\alpha} \rfloor > 1 - \frac{1}{\alpha}$$

$$\left\{ \frac{k}{\alpha} \right\} > \frac{1}{\beta}$$

$\Leftarrow$ Let $n = \lfloor \frac{k}{\alpha} \rfloor + 1$. Then $n\alpha = \lfloor \frac{k}{\alpha} \rfloor \alpha + \alpha$

$$= \frac{k}{\alpha} \cdot \alpha - \{ \frac{k}{\alpha} \} \alpha + \alpha$$

$$= k + \alpha (1 - \{ \frac{k}{\alpha} \})$$

Then $n\alpha > k + \alpha \cdot (1-1) = k$

$n\alpha < k + \alpha \cdot (1 - \frac{1}{\beta}) \leq k+1$

Thus $\lfloor n\alpha \rfloor = k$

b.

Suppose there is no positive intergen n s.t. $\lfloor n\alpha \rfloor = k$

Then $\{ \frac{k}{\alpha} \} < \frac{1}{\beta}$

From $k > \frac{k}{\alpha}$, we have $\{ k - \frac{k}{\alpha} \} = 1 - \{ \frac{k}{\alpha} \} > \frac{1}{\alpha}$

and. $\{ \frac{k}{\beta} \} > \frac{1}{\alpha}$

Thus there exises positice integer n s.t. $\lfloor n\beta \rfloor = k$

For the other side, it's similar.

13.

b. $g(6,13,25) = 5 \oplus 11 = 101_2 \oplus 1011_2 = 1110_2 = 14_{10}$

$5 \oplus 14$ is not achievable,

$11 \oplus 14 = 5_{10}$

Thus winning move is to $(6,13,19)$

$g(23,56,63) = 22 \oplus 6 = 10110_2 \oplus 110_2 = 10000_2 = 16_{10}$

$22 \oplus 16 = 6_{10}$

$6 \oplus 16 = 22_{10}$

Thus winning moves are $(7,56,63)$ and $(23,40,63)$